Fifth Edition

Essential Mathematics for Economic Analysis

Knut Sydsæter, Peter Hammond
Arne Strøm & Andrés Carvajal

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ESSENTIAL MATHEMATICS FOR
ECONOMIC ANALYSIS

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To Knut Sydsæter (1937–2012), an inspiring mathematics teacher, as well as wonderful friend and colleague, whose vision, hard work, high professional standards, and sense of humour were all essential in creating this book.

—Arne, Peter and Andrés

To Else, my loving and patient wife.

—Arne

To the memory of my parents Elsie (1916–2007) and Fred (1916–2008), my first teachers of Mathematics, basic Economics, and many more important things.

—Peter

To Yeye and Tata, my best ever students of "matemáquinas", who wanted this book to start with "Once upon a time ..."

—Andrés

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PREFACE

Once upon a time there was a sensible straight line who was hopelessly in love with a dot. 'You're the beginning and the end, the hub, the core and the quintessence,' he told her tenderly, but the frivolous dot wasn't a bit interested, for she only had eyes for a wild and unkempt squiggle who never seemed to have anything on his mind at all. All of the line's romantic dreams were in vain, until he discovered . . . angles! Now, with newfound self-expression, he can be anything he wants to be — a square, a triangle, a parallelogram . . . And that's just the beginning!
—Norton Juster (*The Dot and the Line: A Romance in Lower Mathematics* 1963)

I came to the position that mathematical analysis is not one of many ways of doing economic theory: It is the only way. Economic theory is mathematical analysis. Everything else is just pictures and talk.
—R. E. Lucas, Jr. (2001)

Purpose

The subject matter that modern economics students are expected to master makes significant mathematical demands. This is true even of the less technical "applied" literature that students will be expected to read for courses in fields such as public finance, industrial organization, and labour economics, amongst several others. Indeed, the most relevant literature typically presumes familiarity with several important mathematical tools, especially calculus for functions of one and several variables, as well as a basic understanding of multivariable optimization problems with or without constraints. Linear algebra is also used to some extent in economic theory, and a great deal more in econometrics.

The purpose of *Essential Mathematics for Economic Analysis*, therefore, is to help economics students acquire enough mathematical skill to access the literature that is most relevant to their undergraduate study. This should include what some students will need to conduct successfully an undergraduate research project or honours thesis.

As the title suggests, this is a book on *mathematics*, whose material is arranged to allow progressive learning of mathematical topics. That said, we do frequently emphasize economic applications, many of which are listed on the inside front cover. These not only help motivate particular mathematical topics; we also want to help prospective economists acquire mutually reinforcing intuition in both mathematics and economics. Indeed, as the list of examples on the inside front cover suggests, a considerable number of economic concepts and ideas receive some attention.

We emphasize, however, that this is not a book about economics or even about mathematical economics. Students should learn economic theory systematically from other courses, which use other textbooks. We will have succeeded if they can concentrate on the economics in these courses, having already thoroughly mastered the relevant mathematical tools this book presents.

Special Features and Accompanying Material

Virtually all sections of the book conclude with exercises, often quite numerous. There are also many review exercises at the end of each chapter. Solutions to almost all these exercises are provided at the end of the book, sometimes with several steps of the answer laid out.

There are two main sources of supplementary material. The first, for both students and their instructors, is via MyMathLab. Students who have arranged access to this web site for our book will be able to generate a practically unlimited number of additional problems which test how well some of the key ideas presented in the text have been understood. More explanation of this system is offered after this preface. The same web page also has a "student resources" tab with access to a *Solutions Manual* with more extensive answers (or, in the case of a few of the most theoretical or difficult problems in the book, the only answers) to problems marked with the special symbol SM.

The second source, for instructors who adopt the book for their course, is an *Instructor's Manual* that may be downloaded from the publisher's Instructor Resource Centre.

In addition, for courses with special needs, there is a brief online appendix on trigonometric functions and complex numbers. This is also available via MyMathLab.

Prerequisites

Experience suggests that it is quite difficult to start a book like this at a level that is really too elementary.[1] These days, in many parts of the world, students who enter college or university and specialize in economics have an enormous range of mathematical backgrounds and aptitudes. These range from, at the low end, a rather shaky command of elementary algebra, up to real facility in the calculus of functions of one variable. Furthermore, for many economics students, it may be some years since their last formal mathematics course. Accordingly, as mathematics becomes increasingly essential for specialist studies in economics, we feel obliged to provide as much quite elementary material as is reasonably possible. Our aim here is to give those with weaker mathematical backgrounds the chance to get started, and even to acquire a little confidence with some easy problems they can really solve on their own.

[1] In a recent test for 120 first-year students intending to take an elementary economics course, there were 35 different answers to the problem of expanding $(a + 2b)^2$.

To help instructors judge how much of the elementary material students really know before starting a course, the *Instructor's Manual* provides some diagnostic test material. Although each instructor will obviously want to adjust the starting point and pace of a course to match the students' abilities, it is perhaps even more important that each individual student appreciates his or her own strengths and weaknesses, and receives some help and guidance in overcoming any of the latter. This makes it quite likely that weaker students will benefit significantly from the opportunity to work through the early more elementary chapters, even if they may not be part of the course itself.

As for our economic discussions, students should find it easier to understand them if they already have a certain very rudimentary background in economics. Nevertheless, the text has often been used to teach mathematics for economics to students who are studying elementary economics at the same time. Nor do we see any reason why this material cannot be mastered by students interested in economics before they have begun studying the subject in a formal university course.

Topics Covered

After the introductory material in Chapters 1 to 3, a fairly leisurely treatment of single-variable differential calculus is contained in Chapters 4 to 8. This is followed by integration in Chapter 9, and by the application to interest rates and present values in Chapter 10. This may be as far as some elementary courses will go. Students who already have a thorough grounding in single-variable calculus, however, may only need to go fairly quickly over some special topics in these chapters such as elasticity and conditions for global optimization that are often not thoroughly covered in standard calculus courses.

We have already suggested the importance for budding economists of multivariable calculus (Chapters 11 and 12), of optimization theory with and without constraints (Chapters 13 and 14), and of the algebra of matrices and determinants (Chapters 15 and 16). These six chapters in some sense represent the heart of the book, on which students with a thorough grounding in single-variable calculus can probably afford to concentrate. In addition, several instructors who have used previous editions report that they like to teach the elementary theory of linear programming, which is therefore covered in Chapter 17.

The ordering of the chapters is fairly logical, with each chapter building on material in previous chapters. The main exception concerns Chapters 15 and 16 on linear algebra, as well as Chapter 17 on linear programming, most of which could be fitted in almost anywhere after Chapter 3. Indeed, some instructors may reasonably prefer to cover some concepts of linear algebra before moving on to multivariable calculus, or to cover linear programming before multivariable optimization with inequality constraints.

Satisfying Diverse Requirements

The less ambitious student can concentrate on learning the key concepts and techniques of each chapter. Often, these appear boxed and/or in colour, in order to emphasize their importance. Problems are essential to the learning process, and the easier ones should definitely be attempted. These basics should provide enough mathematical background for the student to be able to understand much of the economic theory that is embodied in applied work at the advanced undergraduate level.

Students who are more ambitious, or who are led on by more demanding teachers, can try the more difficult problems. They can also study the material in smaller print. The latter is intended to encourage students to ask why a result is true, or why a problem should be tackled in a particular way. If more readers gain at least a little additional mathematical insight from working through these parts of our book, so much the better.

The most able students, especially those intending to undertake postgraduate study in economics or some related subject, will benefit from a fuller explanation of some topics than we have been able to provide here. On a few occasions, therefore, we take the liberty of referring to our more advanced companion volume, *Further Mathematics for Economic Analysis* (usually abbreviated to FMEA). This is written jointly with our colleague Atle Seierstad in Oslo. In particular, FMEA offers a proper treatment of topics like second-order conditions for optimization, and the concavity or convexity of functions of more than two variables—topics that we think go rather beyond what is really "essential" for all economics students.

Changes in the Fourth Edition

We have been gratified by the number of students and their instructors from many parts of the world who appear to have found the first three editions useful.[2] We have accordingly been encouraged to revise the text thoroughly once again. There are numerous minor changes and improvements, including the following in particular:

1. The main new feature is MyMathLab Global,[3] explained on the page after this preface, as well as on the back cover.
2. New exercises have been added for each chapter.
3. Some of the figures have been improved.

Changes in the Fifth Edition

The most significant change in this edition is that, tragically, we have lost the main author and instigator of this project. Our good friend and colleague Knut Sydsæter died suddenly on 29th September 2012, while on holiday in Spain with his wife Malinka Staneva, a few days before his 75th birthday.

The Department of Economics at the University of Oslo has a web page devoted to Knut and his memory.[4] There is a link there to an obituary written by Jens Stoltenberg, at that

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[2] Different English versions of this book have been translated into Albanian, French, German, Hungarian, Italian, Portuguese, Spanish, and Turkish.
[3] Superseded by MyMathLab for this fifth edition.
[4] See http://www.sv.uio.no/econ/om/aktuelt/aktuelle-saker/sydsaeter.html.

time the Prime Minister of Norway, which includes this tribute to Knut's skills as one of his teachers:

> With a small sheet of paper as his manuscript he introduced me and generations of other economics students to mathematics as a tool in the subject of economics. With professional weight, commitment, and humour, he was both a demanding and an inspiring lecturer. He opened the door into the world of mathematics. He showed that mathematics is a language that makes it possible to explain complicated relationships in a simple manner.

There one can also find Peter's own tribute to Knut, with some recollections of how previous editions of this book came to be written.

Despite losing Knut as its main author, it was clear that this book needed to be kept alive, following desires that Knut himself had often expressed while he was still with us. Fortunately, it had already been agreed that the team of co-authors should be joined by Andrés Carvajal, a former colleague of Peter's at Warwick who, at the time of writing, has just joined the University of California at Davis. He had already produced a new Spanish version of the previous edition of this book; he has now become a co-author of this latest English version. It is largely on his initiative that we have taken the important step of extensively rearranging the material in the first three chapters in a more logical order, with set theory now coming first.

The other main change is one that we hope is invisible to the reader. Previous editions had been produced using the "plain TeX" typesetting system that dates back to the 1980s, along with some ingenious macros that Arne had devised in collaboration with Arve Michaelsen of the Norwegian typesetting firm Matematisk Sats. For technical reasons we decided that the new edition had to be produced using the enrichment of plain TeX called LaTeX that has by now become the accepted international standard for typesetting mathematical material. We have therefore attempted to adapt and extend some standard LaTeX packages in order to preserve as many good features as possible of our previous editions.

Other Acknowledgements

Over the years we have received help from so many colleagues, lecturers at other institutions, and students, that it is impractical to mention them all.

At the time when we began revising the textbook, Andrés Carvajal was visiting the Fundaçao Getulio Vargas in Brazil. He was able to arrange assistance from Cristina Maria Igreja, who knows both TeX and LaTeX from her typesetting work for Brazil's most prestigious academic economics journal, the *Revista Brasileira de Economia*. Her help did much to expedite the essential conversion from plain TeX to LaTeX of the computer files used to produce the book.

In the fourth edition of this book, we gratefully acknowledged the encouragement and assistance of Kate Brewin at Pearson. While we still felt Kate's welcome support in the background, our more immediate contact for this edition was Caitlin Lisle, who is Editor for Business and Economics in the Higher Education Division of Pearson. She was always very helpful and attentive in answering our frequent e-mails in a friendly and encouraging way,

and in making sure that this new edition really is getting into print in a timely manner. Many thanks also to Carole Drummond, Helen MacFadyen, and others associated with Pearson's editing team, for facilitating the process of transforming our often imperfect LaTeX files into the well designed book you are now reading.

On the more academic side, very special thanks go to Prof. Dr Fred Böker at the University of Göttingen. He is not only responsible for translating several previous editions of this book into German, but has also shown exceptional diligence in paying close attention to the mathematical details of what he was translating. We appreciate the resulting large number of valuable suggestions for improvements and corrections that he has continued to provide, sometimes at the instigation of Dr Egle Tafenau, who was also using the German version of our textbook in her teaching.

To these and all the many unnamed persons and institutions who have helped us make this text possible, including some whose anonymous comments on earlier editions were forwarded to us by the publisher, we would like to express our deep appreciation and gratitude. We hope that all those who have assisted us may find the resulting product of benefit to their students. This, we can surely agree, is all that really matters in the end.

Andrés Carvajal, Peter Hammond, and Arne Strøm

Davis, Coventry, and Oslo, February 2016

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1 ESSENTIALS OF LOGIC AND SET THEORY

Everything should be made as simple as possible, but not simpler.
—Albert Einstein[1]

Arguments in mathematics require tight logical reasoning; arguments in economic analysis are no exception to this rule. We therefore present some basic concepts from logic. A brief section on mathematical proofs might be useful for more ambitious students.

A short introduction to set theory precedes this. This is useful not just for its importance in mathematics, but also because of the role sets play in economics: in most economics models, it is assumed that, following some specific criterion, economic agents are to choose, optimally, from a feasible *set* of alternatives.

The chapter winds up with a discussion of mathematical induction. Very occasionally, this is used directly in economic arguments; more often, it is needed to understand mathematical results which economists often use.

1.1 Essentials of Set Theory

In daily life, we constantly group together objects of the same kind. For instance, we refer to the faculty of a university to signify all the members of the academic staff. A garden refers to all the plants that are growing in it. We talk about all Scottish firms with more than 300 employees, all taxpayers in Germany who earned between €50 000 and €100 000 in 2004. In all these cases, we have a collection of objects viewed as a whole. In mathematics, such a collection is called a *set*, and its objects are called its *elements*, or its *members*.

How is a set specified? The simplest method is to list its members, in any order, between the two braces { and }. An example is the set $S = \{a, b, c\}$ whose members are the first three letters in the English alphabet. Or it might be a set consisting of three members represented by the letters a, b, and c. For example, if $a = 0$, $b = 1$, and $c = 2$, then $S = \{0, 1, 2\}$. Also,

[1] Attributed; circa 1933.

$S = \{a, b, c\}$ denotes the set of roots of the cubic equation $(x - a)(x - b)(x - c) = 0$ in the unknown x, where a, b, and c are any three real numbers.

Two sets A and B are considered *equal* if each element of A is an element of B and each element of B is an element of A. In this case, we write $A = B$. This means that the two sets consist of exactly the same elements. Consequently, $\{1, 2, 3\} = \{3, 2, 1\}$, because the order in which the elements are listed has no significance; and $\{1, 1, 2, 3\} = \{1, 2, 3\}$, because a set is not changed if some elements are listed more than once.

Alternatively, suppose that you are to eat a meal at a restaurant that offers a choice of several main dishes. Four choices might be feasible—fish, pasta, omelette, and chicken. Then the *feasible set*, F, has these four members, and is fully specified as

$$F = \{\text{fish, pasta, omelette, chicken}\}$$

Notice that the order in which the dishes are listed does not matter. The feasible set remains the same even if the order of the items on the menu is changed.

The symbol "$\varnothing$" denotes the set that has no elements. It is called the *empty set*.[2]

Specifying a Property

Not every set can be defined by listing all its members, however. For one thing, some sets are infinite—that is, they contain infinitely many members. Such infinite sets are rather common in economics. Take, for instance, the *budget set* that arises in consumer theory. Suppose there are two goods with quantities denoted by x and y. Suppose one unit of these goods can be bought at prices p and q, respectively. A consumption bundle (x, y) is a pair of quantities of the two goods. Its value at prices p and q is $px + qy$. Suppose that a consumer has an amount m to spend on the two goods. Then the *budget constraint* is $px + qy \le m$ (assuming that the consumer is free to underspend). If one also accepts that the quantity consumed of each good must be nonnegative, then the *budget set*, which will be denoted by B, consists of those consumption bundles (x, y) satisfying the three inequalities $px + qy \le m$, $x \ge 0$, and $y \ge 0$. (The set B is shown in Fig. 4.4.12.) Standard notation for such a set is

$$B = \{(x, y) : px + qy \le m,\ x \ge 0,\ y \ge 0\} \tag{1.1.1}$$

The braces { } are still used to denote "the set consisting of". However, instead of listing all the members, which is impossible for the infinite set of points in the triangular budget set B, the specification of the set B is given in two parts. To the left of the colon, (x, y) is used to denote the typical member of B, here a consumption bundle that is specified by listing the respective quantities of the two goods. To the right of the colon, the three properties that these typical members must satisfy are all listed, and the set thereby specified. This is an example of the general specification:

$$S = \{\text{typical member : defining properties}\}$$

[2] Note that it is *the*, and not *an*, empty set. This is so, following the principle that a set is completely defined by its elements: there can only be one set that contains no elements. The empty set is the same, whether it is being studied by a child in elementary school or a physicist at CERN—or, indeed, by an economics student in her math courses!

Note that it is not just infinite sets that can be specified by properties—finite sets can also be specified in this way. Indeed, some finite sets almost *have* to be specified in this way, such as the set of all human beings currently alive.

Set Membership

As we stated earlier, sets contain members or elements. There is some convenient standard notation that denotes the relation between a set and its members. First,

$$x \in S$$

indicates that x *is an element of* S. Note the special "belongs to" symbol $\in$ (which is a variant of the Greek letter ε, or "epsilon").

To express the fact that x is *not* a member of S, we write $x \notin S$. For example, $d \notin \{a, b, c\}$ says that d is not an element of the set $\{a, b, c\}$.

For additional illustrations of set membership notation, let us return to the main dish example. Confronted with the choice from the set $F = \{\text{fish, pasta, omelette, chicken}\}$, let s denote your actual selection. Then, of course, $s \in F$. This is what we mean by "feasible set"—it is possible only to choose some member of that set but nothing outside it.

Let A and B be any two sets. Then A is a *subset* of B if it is true that every member of A is also a member of B. Then we write $A \subseteq B$. In particular, $A \subseteq A$. From the definitions we see that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Set Operations

Sets can be combined in many different ways. Especially important are three operations: *union*, *intersection*, and the *difference* of sets, as shown in Table 1.1.

Table 1.1 Elementary set operations

Notation	Name	The set that consists of:
$A \cup B$	*A union B*	The elements that belong to at least one of the sets A and B
$A \cap B$	*A intersection B*	The elements that belong to both A and B
$A \setminus B$	*A minus B*	The elements that belong to set A, but not to B

Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

EXAMPLE 1.1.1 Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 6\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.[3]

Solution: $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3\}$, $A \setminus B = \{1, 2, 4, 5\}$, $B \setminus A = \{6\}$.

[3] Here and throughout the book, we strongly suggest that when reading an example, you first attempt to solve the problem, while covering the solution, and then gradually reveal the proposed solution to see if you are right.

An economic example can be obtained by considering workers in Utopia in 2001. Let A be the set of all those workers who had an income of at least 15 000 Utopian dollars and let B be the set of all who had a net worth of at least 150 000 dollars. Then $A \cup B$ would be those workers who earned at least 15 000 dollars or who had a net worth of at least 150 000 dollars, whereas $A \cap B$ are those workers who earned at least 15 000 dollars and who also had a net worth of at least 150 000 dollars. Finally, $A \setminus B$ would be those who earned at least 15 000 dollars but who had less than 150 000 dollars in net worth.

If two sets A and B have no elements in common, they are said to be *disjoint*. Thus, the sets A and B are disjoint if and only if $A \cap B = \varnothing$.

A collection of sets is often referred to as a *family* of sets. When considering a certain family of sets, it is often natural to think of each set in the family as a subset of one particular fixed set Ω, hereafter called the *universal set*. In the previous example, the set of all Utopian workers in 2001 would be an obvious choice for a universal set.

If A is a subset of the universal set Ω, then according to the definition of difference, $\Omega \setminus A$ is the set of elements of Ω that are not in A. This set is called the *complement* of A in Ω and is sometimes denoted by A^c, so that $A^c = \Omega \setminus A$.[4] When finding the complement of a set, it is *very* important to be clear about which universal set Ω is being used.

EXAMPLE 1.1.2 Let the universal set Ω be the set of all students at a particular university. Moreover, let F denote the set of female students, M the set of all mathematics students, C the set of students in the university choir, B the set of all biology students, and T the set of all tennis players. Describe the members of the following sets: $\Omega \setminus M$, $M \cup C$, $F \cap T$, $M \setminus (B \cap T)$, and $(M \setminus B) \cup (M \setminus T)$.

Solution: $\Omega \setminus M$ consists of those students who are not studying mathematics, $M \cup C$ of those students who study mathematics and/or are in the choir. The set $F \cap T$ consists of those female students who play tennis. The set $M \setminus (B \cap T)$ has those mathematics students who do not both study biology and play tennis. Finally, the last set $(M \setminus B) \cup (M \setminus T)$ has those students who either are mathematics students not studying biology or mathematics students who do not play tennis. Do you see that the last two sets are equal?[5]

Venn Diagrams

When considering the relationships between several sets, it is instructive and extremely helpful to represent each set by a region in a plane. The region is drawn so that all the elements belonging to a certain set are contained within some closed region of the plane. Diagrams constructed in this manner are called *Venn diagrams*. The definitions discussed in the previous section can be illustrated as in Fig. 1.1.1.

By using the definitions directly, or by illustrating sets with Venn diagrams, one can derive formulas that are universally valid regardless of which sets are being considered. For example, the formula $A \cap B = B \cap A$ follows immediately from the definition of the

[4] Other ways of denoting the complement of A include $\complement A$ and $\tilde{A}$.

[5] For arbitrary sets M, B, and T, it is true that $(M \setminus B) \cup (M \setminus T) = M \setminus (B \cap T)$. It will be easier to verify this equality after you have read the following discussion of Venn diagrams.

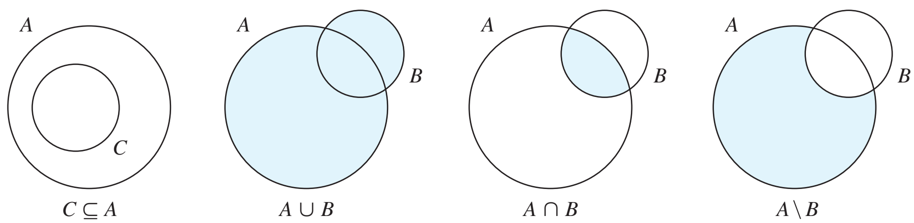

Figure 1.1.1 Venn diagrams

intersection between two sets. It is somewhat more difficult to verify directly from the definitions that the following relationship is valid for all sets A, B, and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{$*$}$$

With the use of a Venn diagram, however, we easily see that the sets on the right- and left-hand sides of the equality sign both represent the shaded set in Fig. 1.1.2. The equality in $(*)$ is therefore valid.

It is important that the three sets A, B, and C in a Venn diagram be drawn in such a way that all possible relations between an element and each of the three sets are represented. In other words, as in Fig. 1.1.3, the following eight different sets all should be nonempty:

1. $(A \cap B) \setminus C$
2. $(B \cap C) \setminus A$
3. $(C \cap A) \setminus B$
4. $A \setminus (B \cup C)$
5. $B \setminus (C \cup A)$
6. $C \setminus (A \cup B)$
7. $A \cap B \cap C$
8. $(A \cup B \cup C)^c$

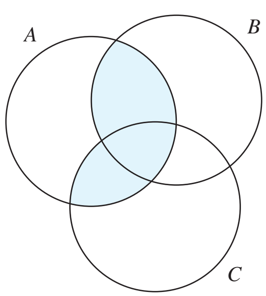

Figure 1.1.2 Venn diagram for $A \cap (B \cup C)$

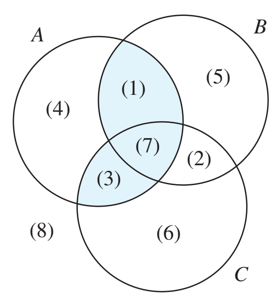

Figure 1.1.3 Venn diagram for three sets

Notice, however, that this way of representing sets in the plane becomes unmanageable if four or more sets are involved, because then there would have to be at least $2^4 = 16$ regions in any such Venn diagram.

From the definition of intersection and union, or by the use of Venn diagrams, it easily follows that $A \cup (B \cup C) = (A \cup B) \cup C$ and that $A \cap (B \cap C) = (A \cap B) \cap C$. Consequently, it does not matter where the parentheses are placed. In such cases, the parentheses can be dropped and the expressions written as $A \cup B \cup C$ and $A \cap B \cap C$. Note, however, that the parentheses cannot generally be moved in the expression $A \cap (B \cup C)$, because this set is not always equal to $(A \cap B) \cup C$. Prove this fact by considering the case where $A = \{1, 2, 3\}$, $B = \{2, 3\}$, and $C = \{4, 5\}$, or by using a Venn diagram.

Cantor

The founder of set theory is Georg Cantor (1845–1918), who was born in St Petersburg but then moved to Germany at the age of eleven. He is regarded as one of history's great mathematicians. This is not because of his contributions to the development of the useful, but relatively trivial, aspects of set theory outlined above. Rather, Cantor is remembered for his profound study of infinite sets. Below we try to give just a hint of his theory's implications.

A collection of individuals are gathering in a room that has a certain number of chairs. How can we find out if there are exactly as many individuals as chairs? One method would be to count the chairs and count the individuals, and then see if they total the same number. Alternatively, we could ask all the individuals to sit down. If they all have a seat to themselves and there are no chairs unoccupied, then there are exactly as many individuals as chairs. In that case each chair corresponds to an individual and each individual corresponds to a chair — i.e., there is a *one-to-one correspondence* between individuals and chairs.

Generally we say that two sets of elements have the same *cardinality*, if there is a one-to-one correspondence between the sets. This definition is also valid for sets with an infinite number of elements. Cantor struggled for three years to prove a surprising consequence of this definition—that there are as many points in a square as there are points on one of the edges of the square, in the sense that the two sets have the same cardinality. In a letter to Richard Dedekind dated 1877, Cantor wrote of this result: "I see it, but I do not believe it."

EXERCISES FOR SECTION 1.1

1. Let $A = \{2, 3, 4\}$, $B = \{2, 5, 6\}$, $C = \{5, 6, 2\}$, and $D = \{6\}$.

(a) Determine which of the following statements are true: $4 \in C$; $5 \in C$; $A \subseteq B$; $D \subseteq C$; $B = C$; and $A = B$.

(b) Find $A \cap B$; $A \cup B$; $A \setminus B$; $B \setminus A$; $(A \cup B) \setminus (A \cap B)$; $A \cup B \cup C \cup D$; $A \cap B \cap C$; and $A \cap B \cap C \cap D$.

2. Let F, M, C, B, and T be the sets in Example 1.1.2.

(a) Describe the following sets: $F \cap B \cap C$, $M \cap F$, and $((M \cap B) \setminus C) \setminus T$.

(b) Write the following statements in set terminology:

(i) All biology students are mathematics students.

(ii) There are female biology students in the university choir.

(iii) No tennis player studies biology.

(iv) Those female students who neither play tennis nor belong to the university choir all study biology.

3. A survey revealed that 50 people liked coffee and 40 liked tea. Both these figures include 35 who liked both coffee and tea. Finally, ten did not like either coffee or tea. How many people in all responded to the survey?

4. Make a complete list of all the different subsets of the set $\{a, b, c\}$. How many are there if the empty set and the set itself are included? Do the same for the set $\{a, b, c, d\}$.

5. Determine which of the following formulas are true. If any formula is false, find a counter example to demonstrate this, using a Venn diagram if you find it helpful.

(a) $A \setminus B = B \setminus A$ (b) $A \cap (B \cup C) \subseteq (A \cap B) \cup C$

(c) $A \cup (B \cap C) \subseteq (A \cup B) \cap C$ (d) $A \setminus (B \setminus C) = (A \setminus B) \setminus C$

6. Use Venn diagrams to prove that: (a) $(A \cup B)^c = A^c \cap B^c$; and (b) $(A \cap B)^c = A^c \cup B^c$

7. If A is a set with a finite number of elements, let $n(A)$ denote its *cardinality*, defined as the number of elements in A. If A and B are arbitrary finite sets, prove the following:

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (b) $n(A \setminus B) = n(A) - n(A \cap B)$

8. A thousand people took part in a survey to reveal which newspaper, A, B, or C, they had read on a certain day. The responses showed that 420 had read A, 316 had read B, and 160 had read C. These figures include 116 who had read both A and B, 100 who had read A and C, and 30 who had read B and C. Finally, all these figures include 16 who had read all three papers.

(a) How many had read A, but not B?

(b) How many had read C, but neither A nor B?

(c) How many had read neither A, B, nor C?

(d) Denote the complete set of all people in the survey by Ω (the universal set). Applying the notation in Exercise 7, we have $n(A) = 420$ and $n(A \cap B \cap C) = 16$, for example. Describe the numbers given in the previous answers using the same notation. Why is $n(\Omega \setminus (A \cup B \cup C)) = n(\Omega) - n(A \cup B \cup C)$?

9. [HARDER] The equalities proved in Exercise 6 are particular cases of the *De Morgan's Laws*. State and prove these two laws:

(a) The complement of the union of any family of sets equals the intersection of all the sets' complements.

(b) The complement of the intersection of any family of sets equals the union of all the sets' complements.

1.2 Some Aspects of Logic

Mathematical models play a critical role in the empirical sciences, especially in modern economics. This has been a useful development in these sciences, but requires practitioners to work with care: errors in mathematical reasoning are easy to make. Here is a typical example of how a faulty attempt to use logic could result in a problem being answered incorrectly.

EXAMPLE 1.2.1 Suppose that we want to find *all* the values of x for which the following equality is true: $x + 2 = \sqrt{4 - x}$.

Squaring each side of the equation gives $(x + 2)^2 = (\sqrt{4 - x})^2$, and thus $x^2 + 4x + 4 = 4 - x$. Rearranging this last equation gives $x^2 + 5x = 0$. Cancelling x results in $x + 5 = 0$, and therefore $x = -5$.

According to this reasoning, the answer should be $x = -5$. Let us check this. For $x = -5$, we have $x + 2 = -3$. Yet $\sqrt{4 - x} = \sqrt{9} = 3$, so this answer is incorrect.[6]

This example highlights the dangers of routine calculation without adequate thought. It may be easier to avoid similar mistakes after studying the structure of logical reasoning.

Propositions

Assertions that are either true or false are called statements, or *propositions*. Most of the propositions in this book are mathematical ones, but other kinds may arise in daily life. "All individuals who breathe are alive" is an example of a true proposition, whereas the assertion "all individuals who breathe are healthy" is a false proposition. Note that if the words used to express such an assertion lack precise meaning, it will often be difficult to tell whether it is true or false. For example, the assertion "67 is a large number" is neither true nor false without a precise definition of "large number".

Suppose an assertion, such as "$x^2 - 1 = 0$", includes one or more variables. By substituting various real numbers for the variable x, we can generate many different propositions, some true and some false. For this reason we say that the assertion is an *open proposition*. In fact, the proposition $x^2 - 1 = 0$ happens to be true if $x = 1$ or -1, but not otherwise. Thus, an open proposition is not simply true or false. Instead, it is neither true nor false until we choose a particular value for the variable.

Implications

In order to keep track of each step in a chain of logical reasoning, it often helps to use "implication arrows". Suppose P and Q are two propositions such that whenever P is true, then Q is necessarily true. In this case, we usually write

$$P \Rightarrow Q \tag{$*$}$$

This is read as "P implies Q"; or "if P, then Q"; or "Q is a consequence of P". Other ways of expressing the same implication include "Q if P"; "P only if Q"; and "Q is an implication of P". The symbol $\Rightarrow$ is an *implication arrow*, and it points in the direction of the logical implication.

[6] Note the wisdom of checking your answer whenever you think you have solved an equation. In Example 1.2.4, below, we explain how the error arose.

EXAMPLE 1.2.2 Here are some examples of correct implications:

(a) $x > 2 \Rightarrow x^2 > 4$

(b) $xy = 0 \Rightarrow$ either $x = 0$ or $y = 0$[7]

(c) S is a square $\Rightarrow$ S is a rectangle

(d) She lives in Paris $\Rightarrow$ She lives in France.

In certain cases where the implication (∗) is valid, it may also be possible to draw a logical conclusion in the other direction: $Q \Rightarrow P$. In such cases, we can write both implications together in a single *logical equivalence*:

$$P \Leftrightarrow Q$$

We then say that "P is equivalent to Q". Because we have both "P if Q" and "P only if Q", we also say that "P if and only if Q", which is often written as "P iff Q" for short. Unsurprisingly, the symbol $\Leftrightarrow$ is called an *equivalence arrow*.

In Example 1.2.2, we see that the implication arrow in (b) could be replaced with the equivalence arrow, because it is also true that $x = 0$ or $y = 0$ implies $xy = 0$. Note, however, that no other implication in Example 1.2.2 can be replaced by the equivalence arrow. For even if x^2 is larger than 4, it is not necessarily true that x is larger than 2 (for instance, x might be -3); also, a rectangle is not necessarily a square; and, finally, the fact that a person is in France does not mean that she is in Paris.

EXAMPLE 1.2.3 Here are some examples of correct equivalences:

(a) $(x < -2 \text{ or } x > 2) \Leftrightarrow x^2 > 4$

(b) $xy = 0 \Leftrightarrow (x = 0 \text{ or } y = 0)$

(c) $A \subseteq B \Leftrightarrow (a \in A \Rightarrow a \in B)$

Necessary and Sufficient Conditions

There are other commonly used ways of expressing that proposition P implies proposition Q, or that P is equivalent to Q. Thus, if proposition P implies proposition Q, we state that P is a "sufficient condition" for Q—after all, for Q to be true, it is sufficient that P be true. Accordingly, we know that if P is satisfied, then it is certain that Q is also satisfied. In this case, we say that Q is a "necessary condition" for P, for Q must necessarily be true if P is true. Hence,

> P is a *sufficient condition* for Q means: $P \Rightarrow Q$
> Q is a *necessary condition* for P means: $P \Rightarrow Q$

The corresponding verbal expression for $P \Leftrightarrow Q$ is, simply, that P *is a necessary and sufficient condition for* Q.

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7 It is important to notice that the word "or" in mathematics is *inclusive*, in the sense that the statement "P or Q" allows for the possibility that P and Q are *both* true.